PROBLEMS with Solutions for ISTCiM 2026

Team Part

Problem A.1
The radical of a positive integer n, denoted by $\mathrm{rad}(n)$, is the product of the distinct prime factors of n; for example, $\mathrm{rad}(2025) = \mathrm{rad}(3^4\,5^2) = 15$, and $\mathrm{rad}(2026) = \mathrm{rad}(2 \cdot 1013) = 2026$.
Prove that there exist infinitely many triples (a, b, c) of coprime positive integers such that $a + b = c$ and $c \geqslant \mathrm{rad}(abc)$.
Remark
Alternatively, we could ask for an explicit construction of infinitely many such triples.

Solution to the Problem A.1:
Let n be a fixed positive integer, and let $a = 1$, $b = 3^{2^n} - 1$, and $c = 3^{2^n}$. Then b can be factored as follows

$$\begin{aligned} b &= \left(3^{2^{n-1}} + 1\right)\left(3^{2^{n-1}} - 1\right) \\ &\vdots \\ &= \left(3^{2^{n-1}} + 1\right)\left(3^{2^{n-2}} + 1\right)\cdots\left(3^{2^0} + 1\right)(3 - 1). \end{aligned}$$

This product is divisible by 2^{n+2}, as the term $\left(3^{2^0} + 1\right) = 4$ is divisible by 2^2, and the remaining n factors in the product are all even. Hence,

$$\mathrm{rad}(b) \leqslant \mathrm{rad}\left(2^{n+2}\frac{b}{2^{n+2}}\right) = 2\,\mathrm{rad}\left(\frac{b}{2^{n+2}}\right) \leqslant 2\left(\frac{b}{2^{n+2}}\right).$$

Because $a, b,$ and c are pairwise coprime, the radical is multiplicative. Thus

$$\begin{aligned} \mathrm{rad}(abc) &= \mathrm{rad}(a), \mathrm{rad}(b), \mathrm{rad}(c) \\ &\leqslant 1 \cdot \left(\frac{2b}{2^{n+2}}\right) \cdot 3 \\ &= \frac{3b}{2^{n+1}} < \frac{3c}{2^{n+1}}, \end{aligned}$$

hence $c > \mathrm{rad}(abc)$.
Remark
This problem is related to the Oesterlé–Masser Conjecture*, more commonly known as the* abc Conjecture.

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Solution 2.:
Let n be a fixed positive integer and let $a = 1$, $b = 2^{6n} - 1$, $c = 2^{6n}$. Then

$$\begin{aligned} b &= \left((2^6)^n - 1\right) = (2^6 - 1)\left(2^{6(n-1)} + 2^{6(n-2)} + \cdots + 1\right) \\ &= 9 \cdot 7 \cdot (\ldots), \end{aligned}$$

hence $\mathrm{rad}(b) = 3\,\mathrm{rad}\left(\frac{b}{9}\right) \leqslant \frac{b}{3} < \frac{c}{3}$. Thus

$$\mathrm{rad}(abc) = \mathrm{rad}(a)\ \mathrm{rad}(b)\ \mathrm{rad}(c) < 1 \cdot \frac{b}{3} \cdot 2 < \frac{2}{3}c < c\,.$$

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Problem A.2
Assume that we start with the one-sign string "$\diamond$". At each step the string consists only of copies of the sign $\diamond$, and we apply to each sign individually one of the following replacements: either we delete it, or we replace it by two copies of the sign $\diamond$, or by five copies of the sign $\diamond$. For example, if we have the string "$\diamond\diamond$", we have the following possibilities for the next step: " ", "$\diamond\diamond$", "$\diamond\diamond\diamond\diamond$", "$\diamond\diamond\diamond\diamond\diamond$", "$\diamond\diamond\diamond\diamond\diamond\diamond\diamond$", "$\diamond\diamond\diamond\diamond\diamond\diamond\diamond\diamond\diamond\diamond$".
Let $\diamond^n$ denote a string consisting of exactly n signs $\diamond$, where $n \in \mathbb{N}$. Determine all $n \in \mathbb{N}$ for which $\diamond^n$ can be obtained in a finite number of steps.

Solution to the Problem A.2:

The task naturally splits into two parts: showing that $\diamond^n$ is achievable for all $n \in \mathbb{N}_0 \setminus \{3\}$ and showing that DD^3 cannot be achieved.
We want to use induction over n, but by the nature of the rules, we need the existence of a few A^n for n small.
The case $n = 1$ is just the start, and we obtain the case $n = 0$ if we apply to the string we start with, the rule that deletes the letter. From the start, we obtain $n = 2$ and $n = 5$ by simply applying the corresponding rule. From the statement, we obtain also $n = 4$ (applying the doubling rule to both letters in $\diamond^2$).
Hence, we assume that we can produce each $\diamond^m$, where $m \in \{1, \dots, n\} \setminus \{3\}$ for some $n \geq 5$.
Let us now try to obtain $\diamond^{n+1}$. Suppose we are given $\diamond^m$, and we write $m = r + s + t$, where $r, s, t \in \mathbb{N}_0$. We apply the deleting rule r-times, the doubling rule s-times and the last rule t-times. Thus, we obtain $\diamond^{2s+5t}$, and we want this to be $\diamond^{n+1}$. Hence $2s + 5t = n + 1$.
If $n + 1$ is even, we can choose $s = \frac{n+1}{2}$, $t = 0$, and $r = m - s$. If $\frac{n+1}{2} \neq 3$, we thus obtain what we want from $\diamond^{\frac{n+1}{2}}$. But, when $\frac{n+1}{2} = 3$, we have $n + 1 = 6$. Now, we obtain $\diamond^6$ from $\diamond^4$ by doubling three of the letters and deleting one.
Assume now that $n+1$ is odd and divisible by 5. We can choose $s = \frac{n+1}{5}$, $t = 0$ and $r = m-s$. A problem occurs if $\frac{n+1}{5} = 3$, that is if we are looking at $\diamond^{15}$. However, in this case we can choose $\diamond^4$ and applying three times the rule that leads to five copies and delete one letter.
Next, we assume that $n+1$ is odd and leaves rest 2 when dividing by 5. In this case, we choose $s = \frac{n+1-2}{5}$ and $t = 1$. We can choose $m = s + t$ or, if this equals 3, we choose $m = s + t + 1$, but we have to prove that these terms are smaller or equal than n.
This is equivalent to

$$\frac{n-1}{5} + 1 + 1 \leq n.$$

This holds if and only if

$$\frac{n-1}{5} \leq n - 2$$

Next, we have

$$n - 1 \leq 5n - 10,$$

so

$$9 \leq 4n,$$

which holds for the numbers n that we consider (actually for $n \geq 3$).
If $n + 1$ is odd and leaves rest 4 when dividing by 5, then we choose $m = \frac{n+1-4}{5}$ and $t = 2$. We can choose $m = s + t$ or, if this is equal to 3, we choose $m = s + t + 1$.
We again have to prove that $s + t + 1 \leq n$. This translates into showing that

$$\frac{n-3}{5} + 2 + 1 \leq n.$$

So,

$$\frac{n-3}{5} \leq n - 3,$$

which is the same as

$$n - 3 \leq 5n - 15,$$

and agrees with

$$12 \leq 4n,$$

and we are again fine as $n \geq 3$.
Now, we show that $\diamond^3$ cannot be achieved. Let us assume by contradiction that we can produce it. As we do not start with $\diamond^3$, we obtain it from applying the rules to some $\diamond^m$. Let us write

$m = r + s + t$ for some choice of $r, s, t \in \mathbb{N}_0$. Denoting by r how many times we delete, by s how many times we double, and by t how many times we replace the letter by five copies, we see that

$$3 = 2s + 5t.$$

As $s, t \in \mathbb{N}_0$, we see that $3 \geq 5t$, hence $t = 0$. But then $3 = 2s$, leaving us with a left-hand side, which is odd, while the right-hand side is even, a contradiction. □

Problem C.1

Determine the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{n}{\sqrt{n}}}}$.

Solution to the Problem C.1:

We will first show that $x > (1+\ln x)\ln x$ for $x \geq e^2$. For $x = e^2$, we have $e^2 > 2.5^2 = 6.25 > 6 = (1+\ln e^2)\ln e^2$. The derivatives of both sides are equal to 1 and $\frac{1+2\ln x}{x}$, respectively. Let us now compare the functions x and $1+2\ln x$. As before, for $x = e^2$, we have $e^2 > 2.5^2 = 6.25 > 5 = 1+2\ln e^2$, and their derivatives satisfy the inequality $1 \geq \frac{2}{x}$ for $x \geq e^2 > 2$. Hence,

$$x\ln x < (1+\ln x)(x-\ln x) = x\ln x + x - (1+\ln x)\ln x$$

so the well-known inequality $e^y < \frac{1}{1-y}$ for $0 < y < 1$ (with $y = \frac{\ln x}{x}$) leads to

$$\sqrt[x]{x} = e^{\frac{\ln x}{x}} < \frac{1}{1-\frac{\ln x}{x}} = \frac{x}{x-\ln x} < \frac{1+\ln x}{\ln x} = 1 + \frac{1}{\ln x}.$$

Thus, for $n \geq 8$ (since $8 > e^2$), we have

$$\begin{aligned}\frac{1}{n^{\frac{n}{\sqrt{n}}}} &> \frac{1}{n^{1+\frac{1}{\ln n}}} = \frac{1}{n^{1+\log_n e}} \\ &= \frac{1}{e\cdot n}.\end{aligned}$$

By the comparison test, since the harmonic series diverges, the series in question is also divergent. □

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Solution 2.:

We can rewrite the general term of the series as follows

$$\frac{1}{n^{\frac{n}{\sqrt{n}}}} = \left(\frac{1}{n}\right)^{n^{1/n}} = \left(\frac{1}{n}\right)^{\left(\frac{1}{n}\right)^{-1/n}}.$$

Let us now the asymptotic behaviour of the function $f(x) = x^{x^{-x}}$ as x tends to 0^+. We have

$$\begin{aligned}x^{-x} &= e^{-x\,\ln x} = 1 - x\,\ln x + \frac{1}{2}x^2\ln^2 x - \cdots \\ &= 1 - x\,\ln x + \mathrm{o}(x\,\ln x)\end{aligned}$$

Hence

$$\begin{aligned}f(x) &= x^{x^{-x}} = x^{(1-x\ln x+\mathrm{o}(x\,\ln x))} \\ &= x\,e^{(-x\ln x+\mathrm{o}(x\,\ln x))\ln x} \\ &= x\left(1 - x\ln^2 x + \mathrm{o}(x\,\ln^2 x)\right).\end{aligned}$$

Since $x\,\ln^2 x$ tends to zero as $x \to 0^+$, there must exist a positive x_0 such that $0 < x\,\ln^2 x - \mathrm{o}(x\,\ln^2 x) < \frac{1}{2}$ for $0 < x < x_0$, meaning $f(x) > \frac{x}{2}$ for such x. Thus

$$\frac{1}{n^{\frac{n}{\sqrt{n}}}} = f\left(\frac{1}{n}\right) > \frac{1}{2n}$$

for $n > \frac{1}{x_0}$ (i.e. for sufficiently large n). Hence the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{n}{\sqrt{n}}}}$ diverges to infinity. □

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Problem C.2

Prove the identity $\int_0^\infty \frac{\sin^2 x}{(t+x)^2}\,\mathrm{d}x = \int_0^\infty \frac{e^{-2tx}}{1+x^2}\,\mathrm{d}x$ for all $t \in \mathbb{R}_+$

Solution to the Problem C.2:

Let $F(t)$ and $G(t)$ denote the left-hand side and right-hand side of the equation, respectively. Since the integrands and their partial derivatives with respect to t are continuous and bounded for all $t > 0$, we are justified in differentiating under the integral sign.

$$F'(t) = -2\int_0^\infty \frac{\sin^2 x}{(t+x)^3}\,\mathrm{d}x$$

and

$$\begin{aligned} F''(t) =& 6\int_0^\infty \frac{\sin^2 x}{(t+x)^4}\,\mathrm{d}x = 3\int_0^\infty \frac{1-\cos 2x}{(t+x)^3}\,\mathrm{d}x \\ =& 3\int_0^\infty \frac{\mathrm{d}x}{(t+x)^4} - 3\int_0^\infty \frac{\cos 2x}{(t+x)^4}\,.\mathrm{d}x \end{aligned}$$

Integrating by part gives

$$= \frac{1}{t^3} + \left.\frac{\cos 2x}{(t+x)^3}\right|_0^\infty + 2\int_0^\infty \frac{\sin 2x}{(t+x)^3}\,\mathrm{d}x$$

and integrating by parts once again yields

$$\begin{aligned} =& \left.\frac{\sin 2x}{(t+x)^2}\right|_0^\infty + 2\int_0^\infty \frac{\cos 2x}{(t+x)^2}\,\mathrm{d}x \\ =& 0 - 4\int_0^\infty \frac{1-\cos 2x}{(t+x)^2}\,\mathrm{d}x + 2\int_0^\infty \frac{\mathrm{d}x}{(t+x)^2}\,\mathrm{d}x \\ =& \frac{2}{t} - 4F(t). \end{aligned}$$

On the other hand

$$\begin{aligned} G''(t) =& \int_0^\infty \frac{e^{-2tx}4x^2}{1+x^2}\,\mathrm{d}x = \int_0^\infty e^{-2tx}\,\mathrm{d}x - 4\int_0^\infty \frac{e^{-2tx}}{1+x^2}\,\mathrm{d}x \\ =& \frac{2}{t} - 4G(t)\,. \end{aligned}$$

Hence, both sides satisfy the same second-order linear differential equation $y'' + y - \frac{2}{t} = 0$. Let $H(t) = F(t) - G(t)$; then H satisfies $H'' + 4H = 0$, meaning

$$F(t) - G(t) = A\cos 2t + B\sin 2t\,.$$

As both F and G tend to zero as t goes to infinity we must have $A = B = 0$.Therefore, we conclude that $F(t) = G(t)$ for all $t > 0$. □

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Solution 2.:

Let

$$I(a,b) = \int_0^\infty \frac{\sin^2 x}{(t+x)^2}\,e^{-(t+x)a}\,e^{-(t+x)b}\,\mathrm{d}x.$$

The left-hand side of our equality is $I(0,0)$, and clearly $I(\infty,\infty) = I(0,\infty) = I(\infty,0) = 0$, where the symbol ∞ denotes the limit as the corresponding variable tends to infinity. Thus,

$$I(0,0) = \int_0^\infty \mathrm{d}a \int_0^\infty \frac{\partial^2}{\partial a\,\partial b} I(a,b)\,\mathrm{d}b.$$

We have

$$\frac{\partial^2}{\partial a\,\partial b} I(a,b) = e^{-(a+b)t} \int_0^\infty \sin^2 x\, e^{-(a+b)x}\,\mathrm{d}x \geq \frac{2e^{-(a+b)t}}{(a+b)((a+b)^2+4)}.$$

Hence,

$$\int_0^\infty \frac{\sin^2 x}{(t+x)^2},\mathrm{d}x = I(0,0) = \int_0^\infty \mathrm{d}a \int_0^\infty \frac{2e^{-(a+b)t}}{(a+b)((a+b)^2+4)}\,\mathrm{d}b$$

Applying the substitution $a+b=u$ and $a-b=v$ gives

$$= \int_0^\infty \mathrm{d}u \int_{-u}^{u} \frac{2e^{-tu}}{u(u^2+4)}\,\frac{1}{2}\,\mathrm{d}v = \int_0^\infty \frac{2e^{-tu}}{u^2+4}\,\mathrm{d}u$$

and finally, the substitution $u=2x$ leads to the required formula

$$= \int_0^\infty \frac{e^{-2tx}}{1+x^2}\,\mathrm{d}x.$$

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Problem E.1
Find all continuous functions $f : (0, \infty) \to (0, \infty)$ satisfying

$$f(xf(y)) = y^2 f(x) \qquad (x, y > 0).$$

Solution to the Problem E.1:
Setting $x = 1$ gives $f(f(y)) = y^2 f(1)$. Since $f(1) > 0$, this proves that f is both injective and surjective on $(0, \infty)$: injectivity follows from strict injectivity of $y \mapsto y^2 f(1)$, and surjectivity because its range is all positive numbers.
Putting $y = 1$ in the original equation gives $f(xf(1)) = f(x)$. Injectivity forces $f(1) = 1$, and hence

$$f(f(y)) = y^2.$$

Given any $t > 0$, choose y with $f(y) = t$. Then $f(t) = y^2$, so the original equation becomes

$$f(xt) = f(x)f(t) \qquad (x, t > 0).$$

The continuous function $h(s) = \log f(e^s)$ is additive on $\mathbb{R}$. An additive continuous function has the form $h(s) = as$: first check rational arguments and then pass to real arguments by continuity. Thus $f(x) = x^a$ for some real a.
The identity $f(f(y)) = y^2$ gives $a^2 = 2$. Both resulting functions satisfy the original equation, so the complete answer is

$$f(x) = x^{\sqrt{2}} \quad \text{or} \quad f(x) = x^{-\sqrt{2}}.$$

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Problem E.2
Show that for $f, g\colon [0,1] \to \mathbb{R}$ such that f is non increasing and g is non decreasing, the following inequality holds

$$\int_0^1 f(x)g(x)\,\mathrm{d}x \;\le\; \int_0^1 f(x)\,\mathrm{d}x \int_0^1 g(x)\,\mathrm{d}x\,.$$

When does the equality hold?

Solution to the Problem E.2:
We have

$$(f(s)-f(t))(g(s)-g(t)) \le 0$$

for any $s,t \in [0,1]$, with equality when $s=t$ or one of the left-hand side factors equals zero. This shows

$$f(s)g(s)+f(t)g(t) \le f(s)g(t)+f(t)g(s)\,.$$

Integrating both sides from 0 to 1 with respect to s and t gives

$$\int_0^1 \mathrm{d}t \int_0^1 f(s)g(s)\,\mathrm{d}s + \int_0^1 \mathrm{d}s \int_0^1 f(t)g(t)\,\mathrm{d}t \le \int_0^1 f(t)\,\mathrm{d}t \int_0^1 g(s)\,\mathrm{d}s + \int_0^1 f(s)\,\mathrm{d}s \int_0^1 g(t)\,\mathrm{d}t\,,$$

which leads to the desired inequality.
Equality holds iff the first inequality holds as an equality almost everywhere, which means that one of the functions f or g has to be constant almost everywhere (i.e., constant on a set of measure 1). But for monotonic function, this means that it has to be constant on $(0,1)$.
Remark
The classical Chebyshev Integral Inequality, stated for comonotone functions (both non-increasing or both non-decreasing), yields the opposite bound $\int_0^1 fg\,\mathrm{d}x \ge \int_0^1 f\,\mathrm{d}x \int_0^1 g\,\mathrm{d}x$.

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Problem G.1

Let T be a nondegenerate tetrahedron. For a plane H meeting the interior of T, let H^+ and H^- be its two closed half-spaces, and set

$$b(H) = \min_{F \text{ a face of } T} \frac{\min\{\text{area}(F \cap H^+), \text{area}(F \cap H^-)\}}{\text{area}(F)}.$$

Find the maximum possible value of $b(H)$ and describe every plane attaining it. An empty piece has area zero.

Solution to the Problem G.1:

If some face has no positive-area piece on one side of H, then $b(H) = 0$. Otherwise every face must contain vertices strictly on both sides. This forces two vertices, say A, B, onto one side and the other two, C, D, onto the other; none lies in the plane.

Let the intersection points on AC, AD, BC, BD divide those edges at fractions $x, y, z, w \in (0, 1)$ measured from A, A, B, B, respectively. The four triangular corner pieces cut off on faces ACD, BCD, ABC, ABD have relative areas

$$p_1 = xy, \qquad p_2 = zw, \qquad p_3 = (1-x)(1-z), \qquad p_4 = (1-y)(1-w).$$

These formulas use only that a triangle formed along two sides has area ratio equal to the product of the two side ratios. As $u(1-u) \le 1/4$,

$$p_1 p_2 p_3 p_4 = x(1-x)y(1-y)z(1-z)w(1-w) \le \left(\frac{1}{4}\right)^4.$$

At least one p_i is at most $1/4$, and $b(H) \le \min_i p_i$. Therefore $b(H) \le 1/4$.

For any division of the four vertices into two pairs, the midpoints of the four edges joining the pairs lie in one plane: use the affine function taking values 1 on one pair and -1 on the other, and take its zero set. This plane cuts every face into relative areas $1/4$ and $3/4$, attaining the bound.

Finally, if $b(H) = 1/4$, all $p_i \ge 1/4$. Equality must then hold in the product bound, forcing $x = y = z = w = 1/2$. These are exactly the midpoint planes just described. There are three such planes, one for each pairing. Thus the maximum is $1/4$. □

Problem G.2
Let $A \in \mathcal{M}_{2026}(\mathbb{C})$ be a matrix whose powers satisfy the trace condition

$$\mathrm{Tr}(A^k) = (-1)^{k+1} - 1 \quad \text{for all } 1 \leq k \leq 2026.$$

Let $B, C \in \mathcal{M}_{2026}(\mathbb{C})$ be matrices that mutually commute with each other and with A. Suppose there exist positive integers p and q satisfying $B^p = I_{2026}$ and $C^q = I_{2026}$. Prove that if $\mathrm{Tr}(ABC) = 2026$, then $ABC = I_{2026}$.

Solution to the Problem G.2:
Evaluating the given trace values $s_k = \mathrm{Tr}(A^k) = (-1)^{k+1} - 1$ for $k = 1, \ldots, 2026$ gives

- For odd k, $\mathrm{Tr}(A^k) = 0$.
- For even k, $\mathrm{Tr}(A^k) = -2$.

We apply Newton's identities, which relate the power sums (s_k) to the elementary symmetric polynomials (δ_k) of the characteristic polynomial $f_A(X)$:

$$k\delta_k = \sum_{i=1}^{k} (-1)^{i-1} \delta_{k-i} s_i.$$

Solving this recurrence relation gives $\delta_{2k-1} = 0$ and $\delta_{2k} = 1$ for all $k = 1, 2, \ldots, 1013$. Thus, the characteristic polynomial of A is

$$f_A(X) = X^{2026} + X^{2024} + X^{2022} + \cdots + X^2 + 1.$$

By the Cayley-Hamilton theorem, $f_A(A) = 0$, which implies

$$A^{2026} + A^{2024} + \cdots + A^2 + I_{2026} = 0.$$

Multiplying both sides by $A^2 - I_{2026}$ gives $A^{2028} = I_{2026}$. Consequently, the eigenvalues of A are roots of unity of order 2028 (excluding ± 1, as $f_A(\pm 1) \neq 0$). Most importantly, all eigenvalues of A lie on the unit circle ($|\lambda_i(A)| = 1$). Furthermore, because the roots of $f_A(X) = 0$ are distinct, A is diagonalisable.
Because $A, B,$ and C mutually commute and are diagonalisable, they are simultaneously diagonalisable. Thus, there exists an invertible matrix P such that $P^{-1}AP = D_A$, $P^{-1}BP = D_B$, and $P^{-1}CP = D_C$ are all diagonal matrices. The product matrix ABC is therefore simultaneously diagonalisable, and its diagonal entries are given by the products of the corresponding eigenvalues $z_i = \lambda_i(A)\mu_i(B)\nu_i(C)$ for $i = 1, \ldots, 2026$.
Since $B^p = I$ and $C^q = I$, their eigenvalues are also roots of unity. Thus

$$|\lambda_i(A)| = 1, \quad |\mu_i(B)| = 1, \quad |\nu_i(C)| = 1.$$

Therefore, $|z_i| = |\lambda_i(A)||\mu_i(B)||\nu_i(C)| = 1$ for all i. We are given $\mathrm{Tr}(ABC) = 2026$, which translates to

$$\sum_{i=1}^{2026} z_i = 2026.$$

Applying the triangle inequality gives

$$2026 = \left| \sum_{i=1}^{2026} z_i \right| \leq \sum_{i=1}^{2026} |z_i| = \sum_{i=1}^{2026} 1 = 2026.$$

Since the sum equals its theoretical maximum, equality holds in the triangle inequality, which forces every complex number z_i to equal 1.
Since all eigenvalues of the diagonalisable matrix ABC are equal to 1, its canonical form is the identity matrix I_{2026}. Thus, $ABC = I_{2026}$.
Remark
Such matrices A, B, C exist. From the proof, we know that any matrix A satisfying the trace conditions must have the characteristic polynomial:

$$f_A(X) = X^{2026} + X^{2024} + \cdots + X^2 + 1$$

The roots of this polynomial are the 2028*-th roots of unity, excluding* 1 *and* -1*. Let us call these roots* $\lambda_1, \lambda_2, \ldots, \lambda_{2026}$*. We can define* A *simply as the diagonal matrix of these roots*

$$A = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_{2026} \end{pmatrix}$$

By definition, the sum of the k*-th powers of these roots perfectly satisfies Newton's identities, so this matrix* A *inherently satisfies* $\mathrm{Tr}(A^k) = (-1)^{k+1} - 1$*. Furthermore, because the roots are* 2028*-th roots of unity, we know that* $A^{2028} = I_{2026}$*.*
And for matrices B *and* C *we simply choose* $B = A^{-1}$ *and* $C = I_{2026}$

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Problem P.1
Let p be an odd prime. Choose a p-element subset uniformly from the grid $\{0, 1, \ldots, p-1\}^2$. Find the probability that its centroid has integer coordinates.

Solution to the Problem P.1:
Identify the grid with $G = \mathbb{F}_p^2$ and write $\zeta = e^{2\pi i/p}$. A chosen subset B has an integer centroid exactly when $\sum_{x\in B} x = 0$ in G. For $u \in G$, set

$$F_u(z) = \prod_{x\in G}(1 + z\zeta^{u\cdot x}).$$

The coefficient of z^p is $\sum_{|B|=p} \zeta^{u\cdot\sum_{x\in B} x}$. The elementary identity

$$\frac{1}{p^2}\sum_{u\in G}\zeta^{u\cdot v} = \begin{cases} 1, & v = 0, \\ 0, & v \neq 0 \end{cases}$$

follows by multiplying two finite geometric sums. Therefore the number Z of favourable subsets is

$$Z = \frac{1}{p^2}\sum_{u\in G}[z^p]F_u(z).$$

For $u = 0$, the coefficient is $\binom{p^2}{p}$. For $u \neq 0$, the map $x \mapsto u \cdot x$ takes each value in $\mathbb{F}_p$ exactly p times. Since p is odd,

$$\prod_{j=0}^{p-1}(1 + z\zeta^j) = 1 + z^p, \qquad F_u(z) = (1 + z^p)^p,$$

and its coefficient of z^p is p. Hence

$$Z = \frac{\binom{p^2}{p} + p(p^2-1)}{p^2}, \qquad \mathbb{P} = \frac{\binom{p^2}{p} + p(p^2-1)}{p^2\binom{p^2}{p}}.$$

For example, at $p = 3$ there are 12 favourable subsets out of 84, giving probability $1/7$. □

Problem P.2
Let

$$C = \left\{ \sum_{j=1}^{\infty} \frac{2\varepsilon_j}{3^j} : \varepsilon_j \in \{0,1\} \right\}$$

Construct a partition of C into an uncountable family $\mathcal{S} = \{S_t\}_{t\in\mathcal{T}}$ of disjoint and dense subsets, i.e. such that (*i*) $|\mathcal{T}| > \aleph_0$, (*ii*) $\bigcup\limits_{t\in\mathcal{T}} S_t = C$, (*iii*) $S_t \cap S_r = \varnothing$, and (*iv*) $S_t \subset C = \overline{S_t}$ for all $t, s \in \mathcal{T}$, $t \neq s$.
Remark
The set C is called Cantor's Set.

Solution to the Problem P.2:
First we will construct such splitting of $[0,1] = \bigcup Q_t$. Interval $[0,1)$ forms a topological group with addition modulo 1 and euclidean topology embeded from the real line. The set $Q = \mathbb{Q} \cap [0,1)$ is a normal, dense subgroup.
Let $\{Q_\xi\}_{\xi\in\Xi}$ be a family of cosets of Q, i.e. $\bigcup\limits_{\xi\in\Xi} = [0,1) \setminus Q$ and $Q_\xi = \xi + Q \mod 1$. It is obvious that Ξ is of continuum power. moreover all Q_ξ are dense in $[0,1)$, as additions is continuous. Let $Q_0 = Q \cup \{1\}$, so Q_0 and all Q_ξ split interval $[0,1)$ into an uncountable family of disjoint, dense subsets.
Let now consider the cannonical embedding $\varphi{:}\mathcal{C} \to [0,1]$ giving by $\varphi\left(\sum\limits_{n=1}^{\infty} \frac{\varepsilon_n}{3^n}\right) = \sum\limits_{n=1}^{\infty} \frac{\varepsilon_n}{2^{n+1}}$.
Then φ is continuous, so the family of counterimages of Q_0 and Q_ξ is the desired splitting of $\mathcal{C}$ into an uncountable family of disjoint, dense subsets. □

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Solution 2.:
An explicit construction of such family.
Cantor set $\mathcal{C}$ is a topological group isomorphic to $\{0,1\}^\omega$ (with topology embedded from real line $\mathbb{R}$). The set $\mathcal{S} = \left\{ \sum\limits_{n=1}^{N} \frac{\varepsilon_n}{3^n} \right\}$ of finite sums is a normal subgroup, moreover it is dense in $\mathcal{C}$ (i.e. $\overline{\mathcal{S}} = \mathcal{C}$). Family of its cosets is the desired family of disjoint, dense subsets. □

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