PROBLEMS with Solutions for ISTCiM 2026

Individual Part

Problem A

Let R be a ring with identity, and let $a, b \in R$ be elements such that $ab + ba = 1$ and $a^3 = a$. Prove that $a^2 = 1$.

Solution to the Problem A:

First, we establish a commutation relation between a^2 and b. Multiplying the equation $ab + ba = 1$ by a on the left gives $a^2b + aba = a$. Next, multiplying $ab + ba = 1$ by a on the right gives $aba + ba^2 = a$. Subtracting the second equation from the first gives

$$a^2b - ba^2 = 0,$$

hence

$$a^2b = ba^2.$$

We now simplify the product a^2ba. Using the relation $a^2b = ba^2$ and the given condition $a^3 = a$, we obtain

$$a^2ba = (a^2b)a = (ba^2)a = ba^3 = ba.$$

Finally, we multiply the original equation by a^2 on the left. Starting with $ab + ba = 1$ and multiplying both sides by a^2 gives

$$a^3b + a^2ba = a^2.$$

Using $a^3 = a$ for the first term and our deduced identity $a^2ba = ba$ for the second term, this equation becomes

$$ab + ba = a^2.$$

Since we are given that $ab + ba = 1$, we substitute this into the left-hand side to conclude, that

$$a^2 = 1.$$

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Problem C
Find the limit $\lim\limits_{n\to\infty} \sqrt[n]{e^n-(1+1/n)^{n^2}}$.

Solution to the Problem C:
We have

$$\frac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n} = e^{n^2\ln\left(1+\frac{1}{n}\right)-n} = e^{-\frac{1}{2}+o(1)}$$

as n tends to infinity. Thus

$$\sqrt[n]{e^n-(1+1/n)^{n^2}} = e\left(1-\frac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n}\right)^{\frac{1}{n}} \overset{n\to\infty}{\longrightarrow} e\left(1-e^{-\frac{1}{2}}\right)^0 = e$$

as $1-e^{-\frac{1}{2}}>0$. □

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Problem E
(I) Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x+f(y)) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
(II) Give an example of non-continuous function satisfying this functional equation.

Solution to the Problem E:
(A) The only such functions are $f(x) = 0$ and $f(x) = x + C$ for any real constant C.
First we will show that the image $f(\mathbb{R})$ forms a group under addition (i. e. it is a subgroup of the additive group of real numbers).
The set $f(\mathbb{R})$ is closed under addition: if $a, b \in f(\mathbb{R})$, then $a = f(u)$ and $b = f(v)$ for some $u, v \in \mathbb{R}$, so

$$a + b = f(u) + f(v) = f(u + f(v)) \in f(\mathbb{R}).$$

Moreover we have

$$f(x) = f\left(x - f(x) + f(x)\right) = f\left(x - f(x)\right) + f(x),$$

hence $f\left(x - f(x)\right) = 0$ for all $x \in \mathbb{R}$, so $0 \in f(\mathbb{R})$. On the other hand we have also

$$0 = f\left(x - f(x)\right) = f\left(x - 2f(x) \ + \ f(x)\right) = f\left(x - 2f(x)\right) + f(x),$$

hence $f\left(x - 2f(x)\right)$ is the additive inverse of $f(x)$.
Now we have two possibilities, either the trivial one where $f(\mathbb{R}) = \{0\}$, meaning $f = 0$, or $f(\mathbb{R})$ is infinite and unbounded. As f is continuous, by the Intermediate Value Property (Darboux's Property) $f(\mathbb{R})$ must be an interval, so it has to be equal to the whole real line $\mathbb{R}$. Thus, for all $x \in \mathbb{R}$, there is a specific y_x such that $f(y_x) = x$. Hence

$$f(x) = f(0 + f(y_x)) = f(0) + f(y_x) = C + x$$

where $C = f(0)$.
(B) An example of such function is the floor function $\lfloor x \rfloor$ (called also the integer part or the "Entier" function), which is equal to the highest integer less or equal to x, i. e. $\lfloor x \rfloor \in \mathbb{Z}$ and $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, because

$$\Big\lfloor x + \lfloor y \rfloor \Big\rfloor = \lfloor x \rfloor + \lfloor y \rfloor .$$

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Problem G

Let A be a real $n \times n$ matrix such that $A^2 = 0$ and $A + A^\mathsf{T}$ is orthogonal. Prove that n is even and that there is an orthogonal matrix Q with

$$Q^\mathsf{T} A Q = \begin{pmatrix} 0 & I_{n/2} \\ 0 & 0 \end{pmatrix}.$$

Solution to the Problem G:

The matrix $U = A + A^\mathsf{T}$ is symmetric and orthogonal, so $U^2 = I$. Because $A^2 = (A^\mathsf{T})^2 = 0$, expansion gives

$$AA^\mathsf{T} + A^\mathsf{T}A = I. \tag{2}$$

We already have $\operatorname{Im} A \subseteq \ker A$. If $v \in \ker A$, (2) gives $v = AA^\mathsf{T}v \in \operatorname{Im} A$. Thus

$$V := \ker A = \operatorname{Im} A.$$

Rank–nullity implies $\dim V = n/2$, so n is even.

For $u \in V^\perp = (\operatorname{Im} A)^\perp = \ker A^\mathsf{T}$, (2) gives $A^\mathsf{T}Au = u$. It follows that A is an isometry from $V^\perp$ onto V. Choose an orthonormal basis $e_1, \ldots, e_{n/2}$ of $V^\perp$, and put $v_i = Ae_i$. Then $v_1, \ldots, v_{n/2}$ is an orthonormal basis of V.

In the orthonormal basis $(v_1, \ldots, v_{n/2}, e_1, \ldots, e_{n/2})$, the map A sends each v_i to zero and each e_i to v_i. Its matrix is precisely the required block matrix. Conversely, this block matrix satisfies the two hypotheses, so the description is exact. □

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Problem P

An urn contains balls numbered from 0 to n. Balls are drawn without replacement until the ball numbered 0 appears. What is the expected value of the sum of the numbers on the selected balls?

Solution to the Problem P:

Suppose, that we draw the balls to the end. Then every sequence S has inverted sequence (in the sense of order) S' and every sequence has equal probability. Let $\sigma(S)$ be the sum of numbers before 0 in S. Then $\sigma(S) + \sigma(S') = \sum_1^n i = \frac{n(n+1)}{2}$ so summing up over all permutations we get $\frac{n(n+1)}{4}$. □

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